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QUATERNIONS.

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Trigonometry.—HAMILTON does not give the quaternion equivalents for all the transformations of Trigonometry, and those which he does give are widely scattered through his works. I have found a *resume'* of the subject fruitful in suggesting new relations between the symbols of quaternions.

The following formulæ from the *Elements* will be made use of:—

$$210. \text{ XI. } Sq^2 = (Sq)^2 + (Vq)^2. \quad (1)$$

$$\text{XII. } V.q^2 = 2Sq.Vq. \quad (2)$$

$$\text{XV. } Tq^2 = (Sq)^2 - (Vq)^2. \quad (3)$$

$$\text{XVII. } SU(q)^2 = 2(SUq)^2 - 1. \quad (4)$$

$$\text{XVIII. } Sq'q = Sq'.Sq + S(Vq'.Vq). \quad (5)$$

$$\text{XIX. } Vq'q = Vq'.Sq + Vq.Sq' + V(Vq'.Vq). \quad (6)$$

$$274. \text{ XII. } (TV:S)\sqrt{q} = \sqrt{[(Tq-Sq) \div (Tq+Sq)]}. \quad (7)$$

$$\text{XI. } TV\sqrt{q} = \sqrt{\frac{1}{2}}(Tq-Sq). \quad (8)$$

$$199. \text{ XII. } S\sqrt{q} = \sqrt{\frac{1}{2}}(Tq+Sq). \quad (9)$$

$$205. \quad V\Sigma = \Sigma V, \quad S\Sigma = \Sigma S. \quad (10)$$

$$294. \text{ II. } V.rV\beta a = aS\beta r - \beta Sra. \quad (11)$$

$$210. \text{ XXIX. } Tq' + Tq = T(q' + q) \text{ if } q' = xq. \quad (12)$$

$$S.rV\beta a = r\beta a - rS\beta a + \beta Sra - aS\beta r. \quad (13)$$

We have, for the trigonometrical functions,

$$\sin \angle q = TVUq, \quad \cos \angle q = SUq,$$

$$\sin n \angle q = TVU(q^n), \quad \cos n \angle q = SU(q^n),$$

$$\tan \angle q = (TV:S)q.$$

Putting Uq for q in (1) and taking the tensor,

$$SU(q^2) = (SUq)^2 + (TVUq)^2, \quad (20)$$

$$\cos 2x = \cos^2 x - \sin^2 x; \text{ or from (4)}$$

$$\cos 2x = 2 \cos^2 x - 1.$$

By treating (2) in a similar way we have

$$\sin 2x = 2 \sin x \cos x.$$

$$\text{From (3)} \quad 1 = (SUq)^2 - (VUq)^2, \quad (21)$$

$$\text{or} \quad 1 = \sin^2 x + \cos^2 x.$$

Introducing in (6) the condition that q' and q be complanar, and substituting versors, we have $VUq'q = VUq'SUq + VUqSUq'$.

Taking the tensor of this equation and observing that $VUq' \parallel VUq$, we have, by (12), $TVUq'q = TVUq'SUq + SUq'TVUq$ (22)

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

We have from (5), since

$$\begin{aligned} S(Vq'.Vq) &= -TVq'TVq \cos \angle (Vq':Vq) = -TVq'.TVq, \\ SUq'q &= SUq'SUq - TVUq'TVUq \\ \cos (x+y) &= \cos x \cos y - \sin x \sin y. \end{aligned} \quad (23)$$

Substituting q^{-1} for q in the last two equations,

$$TVUq'q^{-1} = TVUq'SUq - SUq'TVUq, \quad (24)$$

$$SUq'q^{-1} = SUq'SUq + TVUq'TVUq, \quad (25)$$

which gives the values for the sine and cosine of the difference of two angles.

Adding (22) and (24), $TVUq'q + TVUq'q^{-1} = 2SUq'TVUq'S$.

Putting $q'q^{-1} = r$, $q'q = r'$, $q' = \sqrt{(r'r)}$, $q = \sqrt{(rr')}$ and we have

$$\begin{aligned} TVUr' + TVUr &= 2SU\sqrt{(r'r^{-1})}TVU\sqrt{(r'r)}, \\ \sin x + \sin y &= 2\sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y). \end{aligned}$$

Similarly,

$$\begin{aligned} TVUr' - TVUr &= 2SU\sqrt{(r'r)}TVU\sqrt{(r'r^{-1})}, \\ SUr' + SUr &= 2SU\sqrt{(r'r)}SU\sqrt{(r'r^{-1})}, \\ SUr' - SUr &= -2TVU\sqrt{(r'r)}TVU\sqrt{(r'r^{-1})}. \end{aligned}$$

$$\text{From (8)} \quad 2(TVU\sqrt{q})^2 = TUq - SUq,$$

$$2\sin^2 \frac{1}{2}x = 1 - \cos x.$$

$$\text{From (9)} \quad 2(SU\sqrt{q})^2 = TUq + SUq,$$

$$2\cos^2 \frac{1}{2}x = 1 + \cos x.$$

$$(TV:S)q^2 = \frac{2Sq.TVq}{(Sq)^2 + (Vq)^2} = \frac{2TVq}{Sq} \cdot \frac{(Sq)^2}{(Sq)^2 - (TVq)^2} = \frac{2(TV:S)q}{1 - [(TV:S)q]^2},$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

$$(TV:S)q'q = \frac{(TV:S)q + (TV:S)q'}{1 - (TV:S)q'(TV:S)q'}, \quad (TV:S)q'q^{-1} = \frac{(TV:S)q - (TV:S)q'}{1 + (TV:S)q'(TV:S)q'},$$

$$\tan (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

$$(TV:S)q' + (TV:S)q = \frac{TVUq'q}{SUq'SUq}, \quad (TV:S)q' - (TV:S)q = \frac{TVUq'q^{-1}}{SUq'SUq},$$

$$\tan x \pm \tan y = \frac{\sin (x \pm y)}{\cos x \cos y}.$$

$$(TV:S)\sqrt{(r'r)} = \frac{TVUr' + TVUr}{SUr' + SUr}, \quad (TV:S)\sqrt{(r'r^{-1})} = \frac{TVUr' - TVUr}{SUr' + SUr},$$

$$\tan \frac{1}{2}(x \pm y) = \frac{\sin x \pm \sin y}{\cos x + \cos y}.$$

By (6) we have

$$\begin{aligned} V(q''.q'q) &= Vq'qSq'' + Sq'qVq'' + V(Vq'',Vq'q) \\ &= Vq'SqSq'' + VqSq'Sq'' + Sq''V.Vq'Vq + Vq''Sq'Sq \\ &\quad + Vq''S.Vq'Vq + V(Vq''Vq'Sq + Vq''VqSq' + Vq''V.Vq'Vq), \end{aligned}$$

which becomes, by (10) and (11) and by arranging symmetrically,

$$\begin{aligned} Vq''q'q &= VqSq'Sq'' + Vq'Sq''Sq + Vq''Sq'Sq + SqV.Vq''Vq' \\ &\quad - Sq'V.VqVq'' + Sq''V.Vq'Vq + VqS.Vq''Vq' - Vq'S.VqVq'' \\ &\quad + Vq''S.Vq'Vq. \end{aligned} \quad (39)$$

$$\text{By (5) } S(q''q'q) = Sq''Sq'q + S(Vq''.Vq'q) = Sq''Sq'Sq + Sq''S.Vq'Vq \\ + S[Vq''Vq'Sq + Vq''VqSq' + Vq''V(Vq'Vq)], \text{ or by (10)}$$

$$\begin{aligned} \text{and (13), } S.q''q'q &= Sq''Sq'Sq + Vq''Vq'Vq + Sq''S.Vq'Vq - Vq''S.Vq'Vq \\ &\quad + Sq'S.VqVq'' + Vq'S.VqVq'' + SqS.Vq''Vq' - VqS.Vq'Vq' \\ &= Sq''Sq'Sq + Vq''Vq'Vq + q'S.VqVq'' + KqS.Vq''Vq' + Kq''S.Vq'Vq. \end{aligned} \quad (40)$$

Making the quaternions complanar and their tensors equal, and then taking the tensors of the two equations, we have for the sine and cosine of the sum of three angles,

$$\begin{aligned} TVY^*q''q'q &= TVYq''SYq'SYq + SYq''TVYq'SYq + SYq''SYq'TVYq \\ &\quad - TVYq''TVYq'TVYq. \end{aligned} \quad (41)$$

$$\begin{aligned} SYq''q'q &= SYq''SYq'SYq - TVYq''TVYq'SYq \\ &\quad - TVYq''SYq'TVYq - SYq''TVYq'TVYq. \end{aligned} \quad (42)$$

These equations might have been deduced directly from (22) and (23), but the formulæ for non-complanar quaternions, (39) and (40), are not without value.

From (22), (23), (20), (41) and (42), we obtain

$$TVYq''^2 + TVYq'^2 + TVYq'^2 - TVYq''^2q'^2q^2 = 4TVYq''q'TVYq'qTVYqq''; \quad (43)$$

$$SYq''^2 + SYq'^2 + SYq'^2 + SYq''^2q'^2q^2 = 4SYq''q'SYq'qSYqq''; \quad (44)$$

$$-TVYq''^2 + TVYq'^2 + TVYq'^2 + TVYq''^2q'^2q^2 = 4SYq''q'TVYq'qSYqq''; \quad (45)$$

$$-SYq''^2 + SYq'^2 + SYq'^2 - SYq''^2q'^2q^2 = 4TVYq''q'SYq'qTVYqq''. \quad (46)$$

When $\angle q''q'q = 2n\frac{1}{2}\pi$, $TVYq''^2q'^2q^2 = 0$ and $SYq''^2q'^2q^2 = -1$.

The above equations then become

$$\pm 4TVYqTVYq'TVYq'' = TVYq^2 + TVYq'^2 + TVYq''^2,$$

$$\pm 4SYqSYq'SYq'' = SYq^2 + SYq'^2 + SYq''^2 + 1,$$

$$\pm 4SYqSYq'TVYq'' = TVYq^2 + TVYq'^2 - TVYq''^2,$$

$$\pm 4TVYqTVYq'SYq'' = SYq^2 + SYq'^2 - SYq''^2 - 1,$$

the upper sign being taken when n is even, the lower when n is odd.

When $\angle q''q'q = (2n+1)\frac{1}{2}\pi$, $TVYq''^2q'^2q^2 = 0$ and $SYq''^2q'^2q^2 = 1$, and the equations become

$$\pm 4SYqSYq'SYq'' = TVYq^2 + TVYq'^2 + TVYq''^2,$$

$$\pm 4TVYqTVYq'TVYq'' = SYq^2 + SYq'^2 + SYq''^2 - 1,$$

$$\pm 4TVYqTVYq'SYq'' = TVYq^2 + TVYq'^2 - TVYq''^2,$$

$$\pm 4SYqSYq'TVYq'' = SYq^2 + SYq'^2 - SYq''^2 + 1,$$

according as n is even or odd.

*For want of sorts the Greek Upsilon is here used instead of U .—Compositor.